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Lattice of Tripotents in a JBW*-Triple

C. Martin Edwards[1] and Gottfried T. Rüttimann[2]

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The complete lattice of tripotents in a JBW*-triple and the unit ball in its predual are respectively proposed as models for the complete lattice of propositions and for the generalized normal state space of a nonassociative, noncommutative physical system. A subsystem of such a system may be defined in terms of either principal ideals in the complete lattice of propositions or norm-closed faces of the generalized state space. It is shown that the two definitions are equivalent and that each subsystem is associative.

1. INTRODUCTION

In classical probability theory propositions concerning an empirical system are considered to band together to form a Boolean lattice. Quite often this is the complete Boolean lattice of closed and open sets in a hyperstonian space. In nonclassical, or noncommutative, probability theory and accompanying measure theory this Boolean lattice is replaced by an orthomodular lattice or more general orthostructures. In the concrete situation of a W*-algebra or, more generally, a Jordan W*-algebra the propositional structure under consideration is that of the complete lattice of self-adjoint idempotents in the *-algebra. When the W*-algebra is commutative or the Jordan W*-algebra is associative this reverts to the classical model. In recent years interest has grown in what might be described as nonassociative, noncommutative probability theory. The proper generalization of a Jordan W*-algebra appears not to be an algebra or, indeed, any space equipped with a binary multiplication, but a triple system. Since in this case it makes no sense to speak of idempotents, the natural object of interest is the set of tripotents, that is to say, triple idempotents. The study of the structure of the collection of tripotents in a triple system is the subject of this paper.

[1]Queen's College, Oxford, United Kingdom.
[2]University of Berne, Berne, Switzerland.

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The triple system with which we will be concerned is that known as a JBW*-triple. Examples of JBW*-triples include Jordan W*-algebras, Hilbert spaces, and the space of $m \times n$ matrices over the complex field. The first main result shows that the set of tripotents in a JBW*-triple, when supplemented with a largest element, forms a complete lattice. Moreover, the proper principal ideals in the complete lattice are the complete orthomodular lattices of associative probability theory.

A second object of interest in classical probability theory is the collection of probability measures on the Boolean lattice of propositions. For a noncommutative system represented by a W*-algebra or a Jordan W*-algebra B, this may be replaced by the normal state space of B. The set of states of a subsystem is usually thought of as being a norm-closed face of the normal state space of B. In fact, there is an order isomorphism from the complete orthomodular lattice of self-adjoint idempotents in B onto the complete lattice of norm-closed faces of the normal state space of B. Moreover, the norm-closed face corresponding to a self-adjoint idempotent e in B is the normal state space of the W*-algebra or Jordan W*-algebra, the set of self-adjoint tripotents in which is the principal ideal generated by e. It follows that subsystems can equivalently described either by norm-closed faces of the normal state space or by principal ideals in the complete lattice of propositions. The second main result of this paper shows that the complete lattice of tripotents in a JBW*-triple is order isomorphic to the complete lattice of norm-closed faces of what might be thought of as the generalized normal state space of the nonassociative system. Moreover, each such face is the normal state space of the corresponding associative subsystem. Consequently, subsystems again can equivalently be described either by norm-closed faces of the generalized normal state space or by principal ideals in the complete lattice of propositions. Moreover, every subsystem of this nonassociative, noncommutative system is associative.

2. TRIPOTENTS IN A JBW*-TRIPLE

This section is devoted to a statement of the main results. However, it is first necessary to give the definitions and properties of the structures under consideration.

Recall that a partially ordered set $\mathcal{P}$ is said to be a *lattice* if, for each pair (e, f) of elements of $\mathcal{P}$, the supremum $e \vee f$ and the infimum $e \wedge f$ exist with respect to the partial ordering of $\mathcal{P}$. The partially ordered set $\mathcal{P}$ is said to be a *complete lattice* if, for any subset M of $\mathcal{P}$, the supremum $\vee M$ and the infimum $\wedge M$ exist. A complete lattice has a greatest element and a least element, denoted by 1 and 0, respectively. A complete lattice is said to be *atomic* if, for each nonzero element f in $\mathcal{P}$, there exists a minimal nonzero element e in $\mathcal{P}$ majorized by f. A complete lattice together with an anti-order automorphism $e \mapsto e^{\perp}$ on $\mathcal{P}$ such that, for all elements e and f in $\mathcal{P}$, the supremum of e and $e^{\perp}$ is equal to 1, $e^{\perp\perp}$ is equal to e, and, if $e \leq f$, then f is equal to $e \vee (f \wedge e^{\perp})$, is said to be *orthomodular.*

Let V be a complex vector space and let C be a convex subset of V. A convex subset E of C is said to be a *face* of C provided that, if $tx_1 + (1 - t)x_2$ is an element of E, where x_1 and x_2 lie in C and $0 < t < 1$, then x_1 and x_2 lie in E. Let τ be a locally convex Hausdorff topology on V and let C be τ-closed. Let $\mathcal{F}_\tau(C)$ denote the set of all τ-closed faces of C. Both $\emptyset$ and C are elements of $\mathcal{F}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathcal{F}_\tau(C)$ again lies in $\mathcal{F}_\tau(C)$. Hence, with respect to ordering by set inclusion, $\mathcal{F}_\tau(C)$ forms a complete lattice. A subset E of C is said to be a *τ-exposed face* of C if there exists a τ-continuous linear functional f on V and a real number t such that, for all elements x in $C \backslash E$, $\operatorname{Re} f(x)$ is less than t and, for all elements x in E, $\operatorname{Re} f(x)$ is equal to t. Let $\mathcal{E}_\tau(C)$ denote the set of τ-exposed faces of C. Clearly, $\mathcal{E}_\tau(C)$ is contained in $\mathcal{F}_\tau(C)$ and the intersection of a finite number of elements of $\mathcal{E}_\tau(C)$ again lies in $\mathcal{E}_\tau(C)$. Moreover, both $\emptyset$ and C belong to $\mathcal{E}_\tau(C)$. The intersection of an arbitrary family of elements of $\mathcal{E}_\tau(C)$ is said to be a *τ-semiexposed face* of C. Let $\mathcal{S}_\tau(C)$ denote the set of τ-semiexposed faces of C. Clearly $\mathcal{E}_\tau(C)$ is contained in $\mathcal{S}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathcal{S}_\tau(C)$ again lies in $\mathcal{S}_\tau(C)$. Hence, with respect to the ordering by set inclusion $\mathcal{S}_\tau(C)$ forms a complete lattice and the infimum of a family of elements of $\mathcal{S}_\tau(C)$ coincides with its infimum when taken in $\mathcal{F}_\tau(C)$.

When V is a complex Banach space with dual space V^* the abbreviations n and w^* will be used for the norm topology of V and the weak* topology of V^*. For each subset E of the unit ball V_1 in V and F of the unit ball V_1^* of V^* let the subsets E' and $F_\prime$ be defined by

$$E' = \{a \in V_1^* : a(x) = 1 \ \forall x \in E\}, \qquad F_\prime = \{x \in V_1 : a(x) = 1 \ \forall a \in F\}$$

Notice that E lies in $\mathcal{S}_n(V_1)$ if and only if $(E')_\prime$ coincides with E, F lies in $\mathcal{S}_{w^*}(V_1^*)$ if and only if $(F_\prime)'$ coincides with F, and the mappings $E \mapsto E'$ and $F \mapsto F_\prime$ are anti-order isomorphisms between the complete lattices $\mathcal{S}_n(V_1)$ and $\mathcal{S}_{w^*}(V_1^*)$ and are inverses of each other. The reader is referred to Edwards and Rüttimann (1985, 1988) for details.

Recall that a Jordan *-algebra A which is also a complex Banach space such that, for all elements a and b in A, $\|a^*\| = \|a\|$, $\|a \circ b\| \leq \|a\| \cdot \|b\|$, and $\|\{a\ a\ a\}\| = \|a\|^3$, where

$$\{a\ b\ c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c) \qquad (1)$$

is the Jordan triple product on A, is said to be a *Jordan* C^*-algebra (Wright, 1977) or *JB*-algebra* (Youngson, 1978). A JB*-algebra which is the dual of a Banach space is said to be a *Jordan* W^**-algebra* (Edwards, 1980) or a JBW^*-algebra (Youngson, 1978). A JBW*-algebra A always possesses a unit element 1 and has a unique predual A_*. Examples of JB*-algebras are C*-algebras and examples of JBW*-algebras are W*-algebras, in both cases equipped with the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba)$$

The self-adjoint parts of JB*-algebras and JBW*-algebras are said to be *JB-algebras* and *JBW-algebras,* respectively. The set A^+ of elements a in the self-adjoint part A_{sa} of a JBW*-algebra A which are squares of self-adjoint elements in A forms a norm-closed generating cone for A_{sa}. The set of elements x in the predual A_* of the JBW*-algebra A such that, for all elements a in A^+, $x(a)$ is nonnegative and $x(1)$ is equal to one is said to be the *normal state space* of A. For the properties of C*-algebras and W*-algebras the reader is referred to Pedersen (1979) and Sakai (1971) and for the properties of Jordan algebras to Hanche-Olsen and Størmer (1984), Jacobson (1968), Loos (1975), and Neher (1987).

The set $\mathcal{P}(A)$ of self-adjoint idempotents, the *projections*, in a JBW*-algebra A forms a complete orthomodular lattice with respect to the partial ordering defined, for elements e and f in $\mathcal{P}(A)$, by $e \leq f$ if $e \circ f$ is equal to e, and the mapping $e \to e^\perp$, where $e^\perp$ is equal to $1 - e$. The mapping $e \mapsto \{e\}_,$ is an order isomorphism from $\mathcal{P}(A)$ onto the complete lattice of norm-closed faces of the normal state space of A. Moreover, for each element e in $\mathcal{P}(A)$, $\{e\}_,$ is the normal state space of the JBW*-algebra $\{e\ A\ e\}$ and the complete orthomodular lattice of self-adjoint idempotents in eAe is the principal ideal $\{f \in \mathcal{P}(A): f \leq e\}$ in $\mathcal{P}(A)$ (Battaglia, 1990; Edwards, 1980; Edwards and Rüttimann, 1985).

Recall that $\mathcal{P}(A)$ represents the complete lattice of propositions of an associative noncommutative system and the normal state space of A represents the set of normal states of the system. It follows that every subsystem is represented by a sub-JBW*-algebra of A of the form $\{e\ A\ e\}$, for some self-adjoint idempotent e in A.

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable, and satisfies the identity

$$[D(a, b), D(c, d)] = D(\{a\ b\ c\}, d) - D(c, \{d\ a\ b\})$$

where $[\cdot, \cdot]$ denotes the commutator and D is the mapping from $A \times A$ to A defined by

$$D(a, b)c = \{a\ b\ c\}$$

is said to be a *Jordan*-triple*. A subspace B of A is said to be a *subtriple* when $\{B\ B\ B\}$ is contained in B and is said to be an *inner ideal* when $\{B\ A\ B\}$ is contained in B.

When the Jordan *-triple A is also a Banach space such that D is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a, a)$ is Hermitian with nonnegative spectrum and satisfies

$$\|D(a, a)\| = \|a\|^2$$

then A is said to be a *JB*-triple*. A JB*-triple which is the dual of a Banach space is called a *JBW*-triple*. A JBW*-triple A has a unique predual A_*. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras, in both cases equipped with the Jordan triple product (1). The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to Barton and Timoney (1986), Dineen (1986), Friedman and Russo (1986), Horn (1987), Kaup (1983, 1984), Kaup and Upmeier (1976), and Upmeier (1985, 1986).

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\ u\ u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. Let u be a tripotent in the JBW*-triple A. Then, the weak*-continuous conjugate linear operator $Q(u)$ and, for j equal to 0, 1, and 2, the weak*-continuous linear operators $P_j(u)$ are defined by

$$Q(u)a = \{u\ a\ u\}, \qquad P_2(u) = Q(u)^2$$

$$P_1(u) = 2(D(u, u) - Q(u)^2), \qquad P_0(u) = I - 2D(u, u) + Q(u)^2$$

The results of Barton and Timoney (1986), Horn (1987), and Loos (1975) show that $P_j(u)$ is a weak*-continuous projection onto the eigenspace $A_j(u)$ of $D(u, u)$ corresponding to the eigenvalue $j/2$. The corresponding decomposition

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is said to be the *Peirce decomposition of A relative to u* and $A_j(u)$ is said to be the *Peirce j-space of A relative to u*. For j, k, and l equal to 0, 1, or 2, the Peirce j-space $A_j(u)$ is a sub-JBW*-triple such that $\{A_j(u)\ A_k(u)\ A_l(u)\}$ is contained in $A_{j-k+l}(u)$ when $j - k + l$ is equal to 0, 1, or 2, and equal to $\{0\}$ otherwise. Moreover,

$$\{A_2(u)\ A_0(u)\ A\} = \{A_0(u)\ A_2(u)\ A\} = \{0\}$$

and $A_0(u)$ and $A_2(u)$ are inner ideals in A.

The proof of the following result can be extracted from Edwards and Rüttimann (1988), Friedman and Russo (1985), and Loos (1975).

Lemma 2.1. Let A be a JBW*-triple, let u be a tripotent in A, and let $A_2(u)$ be the Peirce 2-space of A relative to u. For elements a and b in $A_2(u)$ define

$$a \circ b = \{a\, u\, b\}, \qquad a^\dagger = \{u\, a\, u\}$$

Then, with respect to the multiplication $(a, b) \mapsto a \circ b$ and involution $a \mapsto a^\dagger$, $A_2(u)$ is a JBW*-algebra with unit u.

Let u and v be elements of $\mathcal{U}(A)$. We write $u \perp v$ when

$$\{u\, u\, v\} = 0$$

and we write $u \leq v$ if

$$\{u\, v\, u\} = u$$

It follows that $\leq$ is an ordering relation and $\perp$ is a symmetric relation on $\mathcal{U}(A)$. Further properties of these two binary relations are summarized below. The proof of the theorem can be found in Battaglia (1991) and Edwards and Rüttimann (1988).

Theorem 2.2. Let A be a JBW*-triple, let $\mathcal{U}(A)$ be the collection of tripotents in A, and let $\perp$ and $\leq$ be the binary relations defined above. Then:

(i) $(\mathcal{U}(A), \leq)$ is a partially ordered set with least element 0.
(ii) For an element u in $\mathcal{U}(A)$, $u \perp u$ if and only if u is equal to 0.
(iii) If u and v are elements of $\mathcal{U}(A)$ such that $u \perp v$, then $u \vee v$ exists in the partially ordered set $(\mathcal{U}(A), \leq)$.
(iv) If u and v are elements of $\mathcal{U}(A)$ such that $u \leq v$, then there exists a unique element w in $\mathcal{U}(A)$ such that $w \perp u$ and $u \vee w$ is equal to v.
(v) If u, v and w are elements of $\mathcal{U}(A)$ such that $u \leq v$ and $v \perp w$, then $u \perp w$.
(vi) Let $(u_j)_{j\in\Lambda}$ be a family of elements of $\mathcal{U}(A)$. Then $\wedge_{j\in\Lambda} u_j$ exists in the partially ordered set $(\mathcal{U}(A), \leq)$.
(vii) Let $(u_j)_{j\in\Lambda}$ be an increasing net in $\mathcal{U}(A)$. Then $\vee_{j\in\Lambda} u_j$ exists in the partially ordered set $(\mathcal{U}(A), \leq)$.
(viii) Let $(u_j)_{j\in\Lambda}$ be a family of elements of $\mathcal{U}(A)$ such that $\vee_{j\in\Lambda} u_j$ exists in the partially ordered set $(\mathcal{U}(A), \leq)$, and let u be an element of $\mathcal{U}(A)$. If, for all j in Λ, $u_j \perp u$, then $\vee_{j\in\Lambda} u_j \perp u$.

The following result is discussed in Krause (1991, 1995).

Corollary 2.3. Under the conditions of Theorem 2.2, $(\mathcal{U}(A), \leq, \perp)$ is a generalized orthomodular partially ordered set [in the sense of Mayet-Ippolito (1991)] and as such admits a faithful embedding into an orthomodular partially ordered set.

Let $(\mathcal{U}(A)^\sim, \leq)$ be the MacNeille completion of $(\mathcal{U}(A), \leq)$, i.e., $\mathcal{U}(A)^\sim$ is the union of the set $\mathcal{U}(A)$ and a point set $\{u_\infty\}$, and the partial ordering $\leq$ is extended to $\mathcal{U}(A)^\sim$ by writing $u \leq u_\infty$ for all elements u in $\mathcal{U}(A)^\sim$. The complete lattice $(\mathcal{U}(A)^\sim, \leq)$ is said to be the *lattice of tripotents* in A. In order to simplify notation the $\leq$ will be suppressed in future references to $(\mathcal{U}(A)^\sim, \leq)$.

The lattice of tripotents in a JBW*-triple can be taken to represent the lattice of propositions of a nonassociative, noncommutative system. Its principal ideals represent the lattices of propositions of certain subsystems. These are described in the next theorem, the proof of which can be found in Edwards and Rüttimann (1988).

Theorem 2.4. Let $\mathcal{U}(A)^\sim$ be the lattice of tripotents in the JBW*-triple A. Then, for each tripotent u in A, the principal ideal $\{v \in \mathcal{U}(A): v \leq u\}$ is a sub-complete lattice which is identical to the complete orthomodular lattice $\mathcal{P}(A_2(u))$ of self-adjoint idempotents in the JBW*-algebra $A_2(u)$.

It follows that the subsystem determined by a tripotent u is the associative system corresponding to the JBW*-algebra $A_2(u)$.

Recall that, for each element u in $\mathcal{U}(A)$, the set $\{u\}_{\prime}$ is a norm-exposed face of $A_{*,1}$. Define $\{u_\infty\}_{\prime}$ to be the set $A_{*,1}$. The following result was proved in Edwards and Rüttimann (1988).

Theorem 2.5. Let A be a JBW*-triple with predual A_*.

(i) The mapping $u \mapsto \{u\}_{\prime}$ is an order isomorphism from the complete lattice $\mathcal{U}(A)^\sim$ of tripotents in A onto the complete lattice $\mathcal{F}_n(A_{*,1})$ of norm-closed faces of the closed unit ball $A_{*,1}$ in A_*.

(ii) For each element u in $\mathcal{U}(A)^\sim$, $\{u\}_{\prime}$ is the normal state space of the JBW*-algebra $A_2(u)$.

The complete lattice of propositions of a nonassociative system is represented by the complete lattice $\mathcal{U}(A)^\sim$ and the generalized normal state space of the system is represented by the unit ball $A_{*,1}$. Principal ideals in $\mathcal{U}(A)^\sim$ and norm-closed faces of $A_{*,1}$ are alternative descriptions of subsystems. The result above shows that the two descriptions are equivalent and that every subsystem is associative.

A further result proved in Edwards and Rüttimann (1988) connects the complete lattice of tripotents in A with the facial structure of the unit ball in A.

Theorem 2.6. Let A be a JBW*-triple with predual A_*. Then the mapping $u \mapsto \{u\}_{\prime}{}'$ is an anti-order isomorphism from $\mathcal{U}(A)^\sim$ onto the complete lattice $\mathcal{F}_{w*}(A_1)$ of weak*-closed faces of the closed unit ball A_1 in A and

$$\{u\}_{\prime}{}' = u + A_0(u)_1$$

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